# Determination of the Linear Acceleration of an Object with Respect to a Set of Arbitrary Fixed Axes of Measurement

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**A geometrical problem is investigated concerning the determination of the components of acceleration of some point in a body with respect to the axes of a fixed coordinate system. The body is considered to be moving in any arbitrary manner. The acceleration is determined from three arbitrary acceleration components which are not coplanar. The problem is solved with the aid of linear transformations employing nonorthogonal matrices.**

1. WITH the advent of navigation by means of inertial guidance systems[1–3] the requirement for precise measurement of the linear acceleration of aircraft and ships has been substantially increased. Inaccuracies in the determination of the linear acceleration of some point fixed in the vehicle or on its stabilized platform[4] and referred to in the future as the acceleration of the body result mainly from two causes. The first error originates during the measurement and transformation of the coordinate of the accelerometer which measures acceleration along the deformed axis of the accelerometer instead of along the axis associated with the body. (The error in transformation originates from the deformation of the elastic element and the angular deviation of the pendulum.) The second error results from the deviation of the axes of measurement from their desired direction in the body which are used to determine the angular position of that body in space. Inaccuracies of the first type are specified by the dynamics of the accelerometers and their readout transducers, as well as by the kinematics of the angular motion of the body. Inaccuracies of the second type have a purely geometrical origin. In this report only the geometrical problem is investigated. Let measurements be made of linear acceleration of a body with respect to three arbitrary non-coplanar axes fixed within the body. It is necessary to find the components of this acceleration with respect to the axes of an inertial system of coordinates, relative to which the acceleration of the body is measured. The solution of such a problem is necessary in order to determine the basis for tolerances on the arrangement of the accelerometers in the vehicle.

2. Let us introduce the following coordinate systems: $OXYZ$, an inertial system, and $Cxyz$, a system associated with the body (navigation system). We shall assume these systems to be right-hand coordinate systems. Point $O$ is fixed; point $C$ is a moving origin, the acceleration of which we shall label the acceleration of the body. Let **a**, **b**, and **c** be unit vectors of the axes of acceleration measurement associated with the body. If in the inertial space the body has not only translational but also rotational motion, the unit vectors are applied to the appropriate points in the body (for example, to the points to which the elastic elements of the accelerometers are attached. However, in accordance with the established problem, assume that we are able to determine the components of acceleration **w** of the body with respect to the directions **a**, **b**, and **c**. That is, we are able to find the projections of the acceleration **w** of the moving origin $C$ on these directions, and we are able to transpose the unit vectors **a**, **b**, and **c** parallel to themselves (in moving space) to the point $C$ where they constitute a trihedron $Cabc$. In the general case the trihedron is nonorthogonal but, according to agreement, is also nondegenerate.

Let $j_a$, $j_b$, and $j_c$ denote the projection of the vector **w** on the axes of measurement $a$, $b$, and $c$, respectively. According to the mechanical sense of measurement of acceleration along any direction, an accelerometer will furnish the magnitude of the perpendicular projection of acceleration in this direction. Consequently, the quantities $j_a$, $j_b$, and $j_c$ are the perpendicular projections of the vector **w**, $j_a = \mathbf{w}\cdot\mathbf{a} \ldots$ For clarity in Fig. 1, $j_a$ and $j_b$ are represented for the case of planar motion.

This figure also shows non-orthogonal projections $w_a$ and $w_b$ of acceleration **w** on the axes of measurement $a$ and $b$, where $\alpha$ and $\beta$ are angles defining the deflection of the axes of acceleration measurement $a$ and $b$ from the corresponding navigation axes $x$ and $y$.

We shall introduce the matrix $l$ of direction cosines between the axes of the trihedrons $abc$ and $xyz$:

$$l = (abc, xyz) = \begin{matrix} \\ (a) \\ (b) \\ (c) \end{matrix} \begin{Vmatrix} (x) & (y) & (z) \\ l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{Vmatrix}$$

$$l_{11} = \cos(a,x) \qquad l_{12} = \cos(a,y) \;\ldots$$

For example, on Fig. 1, $l_{11} = \cos\alpha$; $l_{12} = \sin\alpha$, $l_{21} = \sin\beta$, $l_{22} = \cos\beta$. The matrix $l$ is not orthogonal if the trihedron $abc$ is not right angled. We shall also establish the row matrices $j = \|j_a\, j_b\, j_c\|$ and $w = \|w_a,\, w_b,\, w_c\|$. The latter is the matrix of parallel projections of the acceleration **w** on the axes $a$, $b$, and $c$, respectively ($\mathbf{w} = w_a\mathbf{a} + w_b\mathbf{b} + w_c\mathbf{c}$). In addition $W = \|W_X,\, W_Y,\, W_Z\|$ is the desired projection of the acceleration **w** on the axis of the inertial coordinate system. We shall continue to investigate matrices of direction cosines ($3 \times 3$ square matrices):

$$L = (xyz, XYZ) = \|L_{ij}\|$$

$$M = (abc, abc) = \|M_{ij}\|(ij = 1,2,3)$$

The first is orthogonal and determines the angular position of the body in the inertial space; the cosines $L_{ij}$ are expressed in terms of the Euler angles $\psi$, $\vartheta$, $\gamma$ (Fig. 2) by known formulas.[5,6] The second represents the metric matrix of the trihedron $abc$.

The problem concerning the calculation of the matrix $W$ is solved with the help of algebraic relationships in the systems of oblique- and right-angled trihedrons; these relationships may be obtained from Refs. 7–9. Given a set of trihedrons $x_k, y_k, z_k$ $(k = 1, 2, \ldots n)$, $l_{pq} = (x_p, y_p, z_p, x_q, y_q, z_q)$ $(3 \times 3)$—a matrix of direction cosines between the axes of the $p$th and $q$th trihedrons in this aggregate, and $M_k$, the metric matrix

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of the $k$th trihedron in this aggregate which reduces to the unit matrix $E$ (3 × 3) for the right-angled trihedron. Then[7] we have

$$l_{1n} = \left(\prod_{s=1}^{n-2} l_{s(s+1)} M_{s+1}^{-1}\right) l_{(n-1)n} \tag{1}$$

$$M_p = l_{pq} M_q^{-1} l_{qp}$$
$$l_{pq} = l_{pk} M_k^{-1} l_{kq} \tag{2}$$

**3.** Let us resolve a certain vector **A** with respect to the axes of the trihedrons $x_p$, $y_p$, $z_p$ and $x_q$, $y_q$, $z_q$, which in the general case are not orthogonal.

$$\boldsymbol{A} = \boldsymbol{A}_{px}\boldsymbol{x}_p + \boldsymbol{A}s_y\boldsymbol{y}_p + \boldsymbol{A}_{pz}\boldsymbol{z}_p = \boldsymbol{A}_{qx}\boldsymbol{x}_q + \boldsymbol{A}_{qy}\boldsymbol{y}_q + \boldsymbol{A}_{qz}\boldsymbol{z}_q$$

where $\boldsymbol{x}_p \ldots \boldsymbol{z}_q$ are unit vectors of the respective axes. These vectors are connected by the matrix relationship:[10, 7]

$$\Gamma_p = \Gamma_q \gamma_{qp} \qquad \Gamma_p = \|\boldsymbol{x}_p, \boldsymbol{y}_p, \boldsymbol{z}_p\|$$
$$\Gamma_q = \|\boldsymbol{x}_q, \boldsymbol{y}_q, \boldsymbol{z}_q\|$$
$$\gamma_{qp} = \|\gamma_{ij}^{(qp)}\| \qquad (i,j = 1,2,3)$$

where[7]

$$\gamma_{qp}^{-1} = \gamma_{pq} \qquad \gamma_{qp} = M_q^{-1} l_{qp} \tag{3}$$

The matrix $\gamma_{qp}$ defines the transformation of the components of the vector **A** with respect to the axes $x_p$, $y_p$, and $z_p$ to its components with respect to the axes $x_q$, $y_q$, and $z_q$:

$$A_q = A_p \gamma_{qp}^T$$

where

$$A_p = \|A_{px}, A_{py}, A_{pz}\|$$
$$A_q = \|A_{qx}, A_{qy}, A_{qz}\|$$

and the index $T$ is the symbol for the transposed matrix. Substituting in the last equation the expression for the matrix from Eq. (3) we arrive at the important relationship

$$A_q = A_p l_{pq} (M_q^{-1})^T \tag{4}$$

We shall now express the components of the vector **A** with respect to the axes of an oblique trihedron in terms of its rectangular components resolved along these non-orthogonal directions. By the definition of the projection of a vector on an axis we have $A_p^* = \boldsymbol{A}\Gamma_p$, where $A_p^* = \|A_{p1}, A_{p2}, A_{p3}\|$ is the matrix of the projection (rectangular) of the vector **A** on the axes $x_p$, $y_p$, and $z_p$, that is, $A_{p1} = \mathbf{A}\cdot \boldsymbol{x}_p, \ldots$

But

$$A_p^* = A_p M_p \tag{5}$$

or

$$A_p = A_p^* M_p^{-1}$$

**4.** Knowing only the foregoing relationships, we shall proceed to solve our problem. On the basis of Eq. (4):

$$W = w(abc, XYZ) \tag{6}$$

However, from Eq. (5) it follows that

$$j = wM \tag{7}$$

We shall find the metric matrix $M$ with the help of Eq. (2):

$$M = (abc, xyz)M_p^{-1}(xyz, abc)$$

where $M_p$ is the metric matrix of the trihedron $xyz$ which along with the matrix $M_p^{-1}$ is a unit matrix.

Thus

$$M = ll^T \tag{8}$$

Using expression (1) we will find that

$$(abc, XYZ) = (abc, xyz)\,(xyz, XYZ) = lL \tag{9}$$

By eliminating the matrices $w$, $M$, and $(abc, XYZ)$ from re-

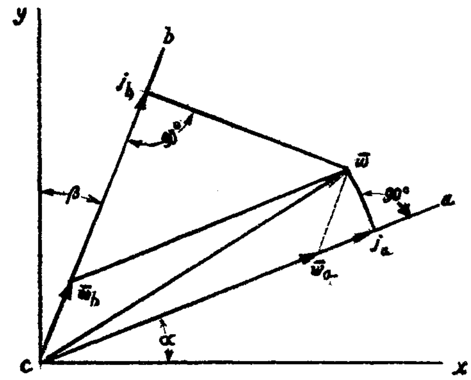


Fig. 1.

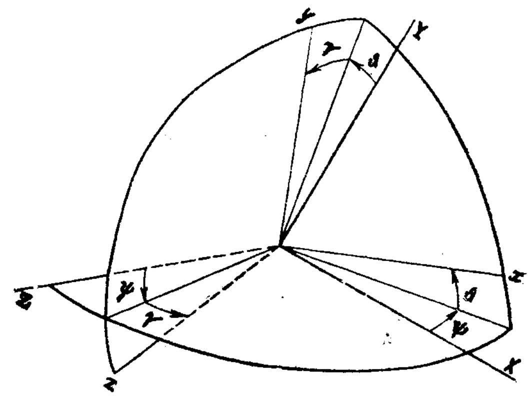

Fig. 2.

lationships (6–9) and taking into account the fact that $w = jM^{-1}$, we obtain the equation

$$W = j(l^T)^{-1} L$$

which expresses the desired projections of the acceleration of a body on the inertial axes in terms of the components of acceleration with respect to the measurement axes of this body. Employing scalars we obtain the computational formulas

$$W_X = \frac{1}{\Lambda}\,[(\Lambda_{11}L_{11} + A_{12}L_{21} + \Lambda_{13}L_{31})j_a + (\Lambda_{21}L_{11} + \Lambda_{22}L_{21} + \Lambda_{23}L_{31})j_b + (\Lambda_{31}L_{11} + \Lambda_{32}L_{21} + \Lambda_{33}L_{31})j_c]$$

$$W_Y = \ldots, \qquad W_Z = \ldots,$$

where $\Lambda_{ij}$ is the cofactor of the element $l_{ij}$, in the determinant $\Lambda = \det l$ ($\Lambda \neq 0$, because, by assumption, the axes $a$, $b$, and $c$ are not coplanar): $\Lambda_{11} = l_{22}l_{33} - l_{32}l_{23}$, etc.

If the axes of measurement coincide with the navigation axes, then $w = j$, $l^T = l^{-1}$, and we will arrive at the obvious relationship

$$W' = jL \qquad (W' = W|_{l=E})$$

With the help of the last equations it is possible to compute the error $\epsilon$ in measurement due to the noncoincidence of the axes of measurement with the navigation axes:

$$\epsilon = W' - W = j[E - (l^T)^{-1}]L$$

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## Reviewer's Comment

The basic objective of this paper is exactly as specified by its title and is succinctly executed, although the notation employed could easily be improved. The problem is a straightforward exercise in linear matrix algebra and transformations between non-orthogonal coordinate systems. Because the usual courses in vector and matrix algebra usually slight this problem and the relationships between parallel and perpendicular projections in such a coordinate system, it should prove interesting for the reader to deduce for himself the stated results [specifically Eqs. (1–5) of the paper].

The results themselves are well known and have been derived and widely used, for example, in the error analysis of attitude-control systems (see, e.g., Ref. 1) and inertial navigation systems. In the latter type of system it is customary, though not mandatory, to place the accelerometers and gyros so that their sensitive axes are aligned along the (orthogonal) navigation axes. Thus any deviations are assumed small, so that small-angle approximations are applicable and the error analysis is considerably simplified, the transformation matrices being then of a very simple nature (see, e.g., Ref. 2).

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# Numerical Solution of the Problem of Supersonic Flow past the Lower Surface of a Delta Wing

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When supersonic flow takes place past a plane delta wing, there is a shock wave upstream of the lower surface. We shall consider cases in which this shock wave is attached to the leading edges of the wing. In this case the flow past the wing will be conical (with center of conicity at the apex of the wing), and the flows past the upper and lower surfaces will be independent. The flow past the upper surface of the wing was investigated in Ref. 1. In the present study we shall solve the problem of flow past the lower surface.

In conical flow all the variables which characterize the flow are constant along lines radiating from a center of conicity. We shall adopt a rectangular system of coordinates associated with the wing, as follows: the origin is at the apex of the wing, at the center of conicity; the $Ox$ axis is directed along the root chord from the apex of the wing toward the trailing edge; the $Oy$ axis is in the plane of the wing, directed toward the right half-wing; the $Oz$ axis is perpendicular to the plane of the wing, directed toward the lower surface.

The position of a straight line emanating from the apex of the wing is determined by the quantities $\eta = y/x$, $\xi = z/x$; thus, all of the flow parameters will be functions of the two variables $\eta$ and $\xi$, which may be considered as dimensionless conical coordinates. The $(\eta,\xi)$ plane corresponds to the physical plane $x = 1$ perpendicular to the root chord of the wing.

We introduce the entropy function

$$\tilde{s} = \frac{1}{\varkappa(\varkappa - 1)} \ln \frac{P/P_\infty}{(\rho/\rho_\infty)^\varkappa}$$

$$u = v_x/V_\infty \qquad v = v_y/V_\infty \qquad w = v_z/V_\infty$$

$$\tilde{a} = a/V_\infty \qquad U^2 = u^2 + v^2 + w^2$$

Here $v_x$, $v_y$, and $v_z$ are the velocity components along the coordinate axes, $V_\infty$ the velocity of the oncoming flow, $a$ the

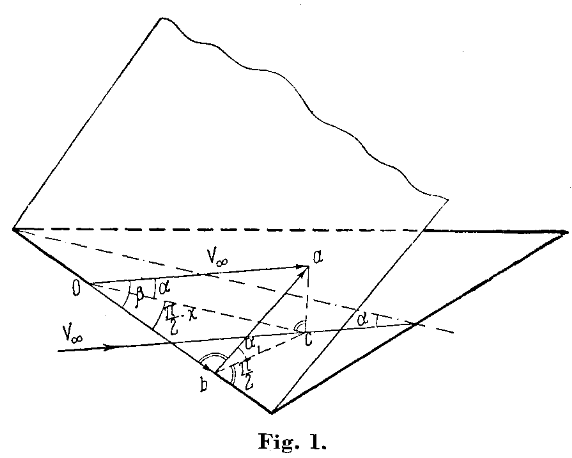

Fig. 1.

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